Kinetic Equations

Solution to the Exercises

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Exercise 1

We controlled in the proof of Theorem 8 (Lecture 6, board 3) the following quantity, defined for $n \in \mathbb{N}$, $n \geqslant 1$:

$$A(n) := 1 + 1 + 2^1 + 2^{1+2} + 2^{1+2+3} + \cdots + 2^{1+2+\cdots+(n-1)} = 1 + \sum_{k=0}^{n-1} 2^{\left(\sum_{j=0}^{k} j\right)}. \tag{1}$$

We introduced then:

$$B(n) := 2^{1+2+\cdots+n} = 2^{\frac{n(n+1)}{2}}. \tag{2}$$

Prove that $A(n) \leqslant B(n)$.

Proof. We prove this by induction. First of all notice that $A(1) = 2 = B(1)$. Moreover for any $n \geqslant 1$ assume that we have $A(n) \leqslant B(n)$; we then get

$$A(n+1) = 1 + \sum_{k=0}^{n} 2^{\left(\sum_{j=0}^{k} j\right)} = 2 + \sum_{k=1}^{n} 2^{\left(\sum_{j=0}^{k} j\right)} = 2 + \sum_{k=0}^{n-1} 2^{\left(\sum_{j=0}^{k+1} j\right)} \tag{3}$$

$$= 2 + \sum_{k=0}^{n-1} 2^{k+1} 2^{\left(\sum_{j=0}^{k} j\right)} \leqslant 2 + 2^n \sum_{k=0}^{n-1} 2^{\left(\sum_{j=0}^{k} j\right)} \leqslant 2 + 2^n A(n) \tag{4}$$

$$\leqslant 2 + 2^n B(n) = 2 + 2^{\frac{n(n+3)}{2}} \leqslant 2^{\frac{n(n+3)}{2}+1} = 2^{\frac{(n+1)(n+2)}{2}} = B(n+1). \tag{5}$$

We can then conclude by induction.

□

Exercise 2

We recall that the homogeneous Boltzmann equation, for radial in the velocity variable solutions f, writes:

$$\partial_t f + L(f) f = J(f), \tag{6}$$

where we saw in the last exercise sheet that L and J can be written as

$$J(f)(t,r) = 16\pi^2 \int_0^{+\infty} \int_0^{+\infty} f(t,u) f(t,v) G(r,u,v) uvdu\, dv, \tag{7}$$

$$G(r,u,v) = \begin{cases} 0 & \text{if } u^2+v^2 \leqslant r^2, \\ 1 & \text{if } u \geqslant r,\ v \geqslant r, \\ \frac{v}{r} & \text{if } u \geqslant r,\ v \leqslant r, \\ \frac{u}{r} & \text{if } u \leqslant r,\ v \geqslant r, \\ \frac{\sqrt{u^2+v^2-r^2}}{r} & \text{if } u^2+v^2 \geqslant r^2,\ u \leqslant r,\ v \leqslant r, \end{cases} \tag{8}$$

$$L(f)(t,r) = \int_0^{+\infty} P(r,r_1) f(t,r_1) r_1^2 dr_1, \tag{9}$$

$$P(r,r_1) = \left(2r + \frac{2r_1^2}{3r}\right) \mathbb{1}_{r_1 \leqslant r} + \left(2r_1 + \frac{2r^2}{3r_1}\right) \mathbb{1}_{r_1 > r}. \tag{10}$$

We now aim at controlling from below the solutions of (6). In order to do so, the first step is to bound from above the term $L(f)$.

We now consider a solution f of (6) on $[0,t_1] \times \mathbb{R}_+$ satisfying that there exist a constant C such that for all $(t,r) \in [0,t_1] \times \mathbb{R}_+$:

$$0 \leqslant f_0(r) \leqslant \frac{C}{(1+r)^k}, \tag{11}$$

$$0 \leqslant f(t,r) \leqslant \frac{C}{r^\alpha (1+r)^\beta}, \tag{12}$$

with $k > 6$, $0 \leqslant \alpha < 3$ and $\alpha + \beta > 6$ fixed.

Prove that there exist two positive real numbers A and B, depending only on the initial datum f_0, such that for all $r \geqslant 0$:

$$L(f)(r) \leqslant Ar + B. \tag{13}$$

Give explicitly the constants A and B.

Hint: Use Theorem 9 to control the solution f only in terms of the initial datum f_0.

Proof. First notice that we can bound $P(r,r_1)$ as

$$P(r,r_1) = \left(2r + \frac{2r_1^2}{3r}\right) \mathbb{1}_{r_1 \leqslant r} + \left(2r_1 + \frac{2r^2}{3r_1}\right) \mathbb{1}_{r_1 > r} \tag{14}$$

$$\leqslant \frac{8}{3} \left(r \mathbb{1}_{r_1 \leqslant r} + r_1 \mathbb{1}_{r < r_1}\right). \tag{15}$$

Using (11) we now get the bound

$$L(f)(r) \leqslant \frac{8}{3} \left[\int_0^r \frac{Cr}{(1+r_1)^k} dr_1 + \int_r^{+\infty} \frac{Cr_1}{(1+r_1)^k} dr_1 \right] \tag{16}$$

$$\leqslant \left(\frac{8}{3} \int_0^{+\infty} \frac{C}{(1+r_1)^k} dr_1 \right) r + \left(\frac{8}{3} \int_0^{+\infty} \frac{Cr_1}{(1+r_1)^k} dr_1 \right). \tag{17}$$

□

Exercise 3

- Let f be a solution of (6) satisfying the assumptions of the previous exercise. Show that for all $(t,r) \in [0,t_1] \times \mathbb{R}_+$:

$$f(t,r) \geqslant f_0(r)\, e^{-(Ar+B)t} + \int_0^t e^{-(Ar+B)(t-s)} J(f(s,r))\, ds, \tag{18}$$

where A and B are the constants found in (13).

This is already a first interesting control on the solutions. Nevertheless, when the initial datum is 0 at r_0, the control (18) on the quantity $f(t,r_0)$ becomes trivial.

- Deduce that for all $(t,r) \in [0,t_1] \times \mathbb{R}_+$:

$$\begin{aligned} f(t,r) \geqslant 16\pi^2 t e^{-(Ar+B)t} \int_0^{+\infty}\int_0^{+\infty} & G(r,u,v)\cdot \\ \cdot f_0(u)\, e^{-(Au+B)t} f_0(v)\, e^{-(Av+B)t} & uv du\ dv. \end{aligned} \tag{19}$$

- Let us now assume that the initial datum f_0 is non-zero, that is there exists $r_0 \in \mathbb{R}_+$ and $d, m > 0$ such that

$$f_0(r) \geqslant m > 0 \tag{20}$$

for all $r \in [r_0, r_0 + d]$. Using the assumption (20), prove that

$$f(t,r) > 0 \tag{21}$$

for all $t > 0$ and r small enough (that is, smaller than a constant which depends only on r_0 and d).

Hint: Consider, for an arbitrary $1 < \gamma < \sqrt{2}$, the bound $b = \min_{u,v \geqslant \frac{r}{\gamma}} G(r,u,v)$, and use this quantity to bound from below the integral in (19).

Proof. From the previous exercise we get that

$$\partial_t f(t,r) = J(f)(t,r) - L(f)(t,r) f(t,r) \geqslant J(f)(t,r) - (Ar+B) f(t,r). \tag{22}$$

We then get

$$e^{(Ar+B)t} f(t,r) = f_0(r) + \int_0^t e^{(Ar+B)s} \left[(Ar+B) f(s,r) + \right. \tag{23}$$

$$\left. + J(f)(s,r) - L(f)(s,r) f(s,r)\right] ds \tag{24}$$

$$\geqslant f_0(r) + \int_0^t e^{(Ar+B)s} J(f)(s,r)\, ds \tag{25}$$

which in turn implies (18). Now, given that f is positive we deduce from (18)

$$f(t,r) \geqslant e^{-(Ar+B)t} f_0(r), \tag{26}$$

$$f(t,r) \geqslant \int_0^t e^{-(Ar+B)(t-s)} J(f)(s,r)\, ds. \tag{27}$$

Using (26) we can bound $J(f)$ from below as

$$J(f)(t,r) \geqslant 16\pi^2 \int_0^{+\infty}\int_0^{+\infty} G(r,u,v)\, f_0(u)\, e^{-(Au+B)t} f_0(v)\, e^{-(Av+B)t} uvdu\ dv, \qquad (28)$$

and now from (27) we get

$$f(t,r) \geqslant 16\pi^2 \int_0^t e^{-(Ar+B)(t-s)} \int_0^{+\infty}\int_0^{+\infty} G(r,u,v)\cdot \qquad (29)$$

$$\cdot f_0(u)\, e^{-(Au+B)s} f_0(v)\, e^{-(Av+B)s} uvdu\ dv\ ds. \qquad (30)$$

Given now that A and B are both positive we have that $e^{(Ar+B)t} \geqslant 1$ and that $e^{-(Ar+B)s} \geqslant e^{-(Ar+B)t}$ and therefore

$$f(t,r) \geqslant 16\pi^2 e^{-(Ar+B)t} t \int_0^{+\infty}\int_0^{+\infty} G(r,u,v)\cdot \qquad (31)$$

$$\cdot f_0(u)\, e^{-(Au+B)t} f_0(v)\, e^{-(Av+B)t} uvdu\ dv. \qquad (32)$$

Finally, assume that $r \leqslant r_0$ and notice that by the definition of G we get

$$f(t,r) \geqslant 16\pi^2 e^{-(Ar+B)t} t \int_r^{+\infty}\int_r^{+\infty} f_0(u)\, e^{-(Au+B)t} f_0(v)\, e^{-(Av+B)t} uvdu\ dv \qquad (33)$$

$$\geqslant 16\pi^2 e^{-(Ar+B)t} t \int_{r_0}^{r_0+d}\int_{r_0}^{r_0+d} f_0(u)\, e^{-(Au+B)t} f_0(v)\, e^{-(Av+B)t} uvdu\ dv \qquad (34)$$

$$\geqslant 16\pi^2 e^{-(Ar+B)t} t m^2 d^2 r_0^2 e^{-2(A(r_0+d)+B)t} > 0. \qquad (35)$$

□